$$i\hbar\partial_t \begin{pmatrix} c_e \\ c_g \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} -\delta & \Omega \\ \Omega^* & \delta \end{pmatrix}\begin{pmatrix} c_e \\ c_g \end{pmatrix}$$

$$\dot c_e = \frac{i}{2}\delta c_e - \frac{i}{2}\Omega c_g \qquad \dot c_e^* = -\frac{i}{2}\delta c_e^* + \frac{i}{2}\Omega^* c_g^*$$

$$\dot c_g = -\frac{i}{2}\Omega^* c_e - \frac{i}{2}\delta c_g \qquad \dot c_g^* = \frac{i}{2}\Omega c_e^* + \frac{i}{2}\delta c_g^*$$

$$\varrho = \begin{pmatrix} \varrho_{ee} & \varrho_{eg} \\ \varrho_{ge} & \varrho_{gg} \end{pmatrix} = \begin{pmatrix} c_e c_e^* & c_e c_g^* \\ c_g c_e^* & c_g c_g^* \end{pmatrix} \qquad \Rightarrow \varrho_{ge} = \varrho_{eg}^*$$

$$\dot\varrho_{gg} = \dot c_g c_g^* + c_g \dot c_g^* = -\frac{i}{2}\Omega^* \underbrace{c_e c_g^*}_{\varrho_{eg}} \cancel{-\frac{i}{2}\delta c_g c_g^*} + \frac{i}{2}\Omega \underbrace{c_e^* c_g}_{\varrho_{ge}} \cancel{+\frac{i}{2}\delta c_g^* c_g} = \frac{i}{2}\left(\Omega\varrho_{ge} - \Omega^*\varrho_{eg}\right)$$

$$\dot\varrho_{ee} = \dot c_e c_e^* + c_e \dot c_e^* = \cancel{\frac{i}{2}\delta c_e c_e^*} - \frac{i}{2}\Omega \underbrace{c_g c_e^*}_{\varrho_{ge}} \cancel{-\frac{i}{2}\delta c_e^* c_e} + \frac{i}{2}\Omega^* \underbrace{c_g^* c_e}_{\varrho_{eg}} = \frac{i}{2}\left(\Omega^*\varrho_{eg} - \Omega\varrho_{ge}\right)$$

$$\dot\varrho_{ge} = \dot c_g c_e^* + c_g \dot c_e^* = -\frac{i}{2}\Omega^* \underbrace{c_e c_e^*}_{\varrho_{ee}} - \frac{i}{2}\delta \underbrace{c_g c_e^*}_{\varrho_{ge}} - \frac{i}{2}\delta \underbrace{c_e^* c_g}_{\varrho_{ge}} + \frac{i}{2}\Omega^* \underbrace{c_g^* c_g}_{\varrho_{gg}} = \frac{i}{2}\Omega^*\left(\varrho_{gg} - \varrho_{ee}\right) - i\delta\varrho_{ge}$$

$$\dot\varrho_{eg} = \dot c_e c_g^* + c_e \dot c_g^* = \frac{i}{2}\delta \underbrace{c_e c_g^*}_{\varrho_{eg}} - \frac{i}{2}\Omega \underbrace{c_g c_g^*}_{\varrho_{gg}} + \frac{i}{2}\Omega \underbrace{c_e^* c_e}_{\varrho_{ee}} + \frac{i}{2}\delta \underbrace{c_g^* c_e}_{\varrho_{eg}} = \frac{i}{2}\Omega\left(\varrho_{ee} - \varrho_{gg}\right) + i\delta\varrho_{eg} = \dot\varrho_{ge} \checkmark$$

Optical Bloch equations in density matrix form

(I) $\dot{\varrho}_{gg} = \frac{i}{2}(\Omega \varrho_{ge} - \Omega^* \varrho_{eg}) + \Gamma \varrho_{ee}$

(II) $\dot{\varrho}_{ee} = \frac{i}{2}(\Omega^* \varrho_{eg} - \Omega \varrho_{ge}) - \Gamma \varrho_{ee} = -\left(\frac{i}{2}(\Omega \varrho_{ge} - \Omega^* \varrho_{eg}) + \Gamma \varrho_{ee}\right) = -\dot{\varrho}_{gg}$

(III) $\dot{\varrho}_{ge} = \frac{i}{2}\Omega^*(\varrho_{gg} - \varrho_{ee}) - \left(\frac{\Gamma}{2} + i\delta\right)\varrho_{ge}$

(IV) $\dot{\varrho}_{eg} = \frac{i}{2}\Omega(\varrho_{ee} - \varrho_{gg}) - \left(\frac{\Gamma}{2} - i\delta\right)\varrho_{eg}$

Translation table from ϱ to R

$$\varrho = \frac{1}{2}(I + \vec{R}\cdot\vec{\sigma}) = \begin{pmatrix} \varrho_{ee} & \varrho_{eg} \\ \varrho_{ge} & \varrho_{gg} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} \; ; \; I = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} ; \sigma_x = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} ; \sigma_y = \begin{pmatrix} & -i \\ i & \end{pmatrix} ; \sigma_z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$\frac{1}{2}(1+z) = \varrho_{ee}$ $\quad z = 2\varrho_{ee} - 1$ $\qquad \frac{1}{2}(x-iy) = \varrho_{eg}$

$\frac{1}{2}(1-z) = \varrho_{gg}$ $\qquad \frac{1}{2}(x+iy) = \varrho_{ge}$

$1 = \varrho_{ee} + \varrho_{gg}$ $\qquad iy = \varrho_{ge} - \varrho_{eg}$ $\qquad y = -i(\varrho_{eg} - \varrho_{ge})$ $\qquad \dot{y} = i(\dot{\varrho}_{eg} - \dot{\varrho}_{ge})$

$z = \varrho_{ee} - \varrho_{gg} =$ $\qquad x = \varrho_{eg} + \varrho_{ge}$ $\qquad \dot{x} = \dot{\varrho}_{eg} + \dot{\varrho}_{ge}$

$\dot{z} = \dot{\varrho}_{ee} - \dot{\varrho}_{gg} = 2\dot{\varrho}_{ee} = -2\dot{\varrho}_{gg}$

$\dot{\varrho}_{ee} = -\dot{\varrho}_{gg}$ ✓ $\Rightarrow$ (I) & (II) consistent

Translation from Ω, Ω^* to Ω_R, Ω_I

$$\Omega = \Omega_R + i\Omega_I$$

$$\Omega^* = \Omega_R - i\Omega_I$$

$$\Omega_R = \frac{1}{2}(\Omega + \Omega^*)$$

$$\Omega_I = -\frac{i}{2}(\Omega - \Omega^*)$$

Translation of opt. Bloch eqn. from density matrix form to Bloch vector form

(II) $\cancel{\frac{1}{2}}\dot z = \cancel{\frac{i}{2}}\left((\Omega_R - i\Omega_I)\frac{1}{2}(x-iy) - (\Omega_R + i\Omega_I)\frac{1}{2}(x+iy)\right) - \Gamma\,\cancel{\frac{1}{2}}(1+z)$

$$\dot z = x\left[\frac{i}{2}(\cancel{\Omega_R} - i\Omega_I) - \frac{i}{2}(\cancel{\Omega_R} + i\Omega_I)\right]$$

$$+\, y\left[+\frac{1}{2}(\Omega_R - \cancel{i\Omega_I}) + \frac{1}{2}(\Omega_R + \cancel{i\Omega_I})\right]$$

$$-\,(1+z)\,\Gamma$$

$$= x\,(+1)\,\Omega_I + y\,(+1)\,\Omega_R - (1+z)\,\Gamma$$

(III) $\frac{1}{2}(\dot x + i\dot y) = -\frac{i}{2}(\Omega_R - i\Omega_I)z - \left(\frac{\Gamma}{2} + i\delta\right)\frac{1}{2}(x + iy)$

(IV) $\frac{1}{2}(\dot x - i\dot y) = +\frac{i}{2}(\Omega_R + i\Omega_I)z - \left(\frac{\Gamma}{2} - i\delta\right)\frac{1}{2}(x - iy)$

(III)+(IV): $\dot x = x\left[-\frac{1}{2}\left(\frac{\Gamma}{2} + \cancel{i\delta}\right) - \frac{1}{2}\left(\frac{\Gamma}{2} - \cancel{i\delta}\right)\right]$

$$+\, y\left[-\frac{i}{2}\left(\cancel{\frac{\Gamma}{2}} + i\delta\right) + \frac{i}{2}\left(\cancel{\frac{\Gamma}{2}} - i\delta\right)\right]$$

$$+\, z\left[-\frac{i}{2}(\cancel{\Omega_R} - i\Omega_I) + \frac{i}{2}(\cancel{\Omega_R} + i\Omega_I)\right]$$

$$= x\left(-\frac{\Gamma}{2}\right) + y\,(\delta) + z\,(-\Omega_I)$$

(III)−(IV): $i\dot y = x\left[-\frac{1}{2}\left(\cancel{\frac{\Gamma}{2}} + i\delta\right) + \frac{1}{2}\left(\cancel{\frac{\Gamma}{2}} - i\delta\right)\right]$ $\Rightarrow \dot y = x(-\delta) + y\left(-\frac{\Gamma}{2}\right) + z\left(-\Omega_R\right)$

$$+\, y\left[-\frac{i}{2}\left(\frac{\Gamma}{2} + \cancel{i\delta}\right) - \frac{i}{2}\left(\frac{\Gamma}{2} - \cancel{i\delta}\right)\right]$$

$$+\, z\left[-\frac{i}{2}(\Omega_R - \cancel{i\Omega_I}) - \frac{i}{2}(\Omega_R + \cancel{i\Omega_I})\right]$$

$$= x\,(-i\delta) + y\left(-i\frac{\Gamma}{2}\right) + z\,(-i\,\Omega_R)$$

Opt Bloch eqns in Bloch vector form

$$\dot{x} = -\frac{\Gamma}{2} x + \delta y - \Omega_I z$$

$$\dot{y} = -\delta x - \frac{\Gamma}{2} y - \Omega_R z$$

$$\dot{z} = - \Omega_I x + \Omega_R y - (1+z)\Gamma$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & \delta & -\Omega_I \\ -\delta & 0 & -\Omega_R \\ \Omega_I & \Omega_R & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \frac{\Gamma}{2} \begin{pmatrix} x \\ y \\ 2(1+z) \end{pmatrix}$$

Check of cross-product form of opt. Bloch eqn.

$$\frac{\partial}{\partial t}\vec{R} = \vec{\Omega}\times\vec{R}$$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \begin{pmatrix}\Omega_y z - \Omega_z y\\ \Omega_z x - \Omega_x z\\ \Omega_x y - \Omega_y x\end{pmatrix} \stackrel{!}{=} \begin{pmatrix}-\Omega_I z + \delta y\\ -\delta x + \Omega_R z\\ \Omega_I x + \Omega_R y\end{pmatrix} \begin{matrix}\rightarrow \Omega_y = -\Omega_I \; ; \; \Omega_z = -\delta\\ \rightarrow \Omega_z = -\delta \; ; \; \Omega_x = +\Omega_R\\ \rightarrow \Omega_x = \Omega_R \; ; \; \Omega_y = -\Omega_I\end{matrix} \quad \checkmark$$

$$\Rightarrow \vec{\Omega} = \begin{pmatrix}\Omega_R\\ -\Omega_I\\ -\delta\end{pmatrix} = \begin{pmatrix}|\Omega|\cos\varphi_L\\ -|\Omega|\sin\varphi_L\\ -\delta\end{pmatrix}$$

$$\Omega = |\Omega|e^{-i\varphi_L} = |\Omega|\cos(-\varphi_L) + i|\Omega|\sin(-\varphi_L) = \underbrace{|\Omega|\cos\varphi_L}_{\Omega_R} + i\underbrace{(-|\Omega|\sin\varphi_L)}_{\Omega_I}$$