

Calculation step from diff. eqn. for  $c_e$  and  $c_g$  to diff. eqn. for density matrix

$$i\hbar \partial_t \begin{pmatrix} c_e \\ c_g \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} -\delta & \mathcal{N} \\ \mathcal{N}^* & \delta \end{pmatrix} \begin{pmatrix} c_e \\ c_g \end{pmatrix}$$

$$\dot{c}_e = -\frac{i}{2} \delta c_e - \frac{i}{2} \mathcal{N} c_g$$

$$\dot{c}_e^* = -\frac{i}{2} \delta c_e^* + \frac{i}{2} \mathcal{N}^* c_g^*$$

$$\dot{c}_g = -\frac{i}{2} \mathcal{N}^* c_e - \frac{i}{2} \delta c_g$$

$$\dot{c}_g^* = \frac{i}{2} \mathcal{N} c_e^* + \frac{i}{2} \delta c_g^*$$

$$\rho = \begin{pmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{pmatrix} = \begin{pmatrix} c_e c_e^* & c_e c_g^* \\ c_g c_e^* & c_g c_g^* \end{pmatrix}$$

$$\Rightarrow \rho_{ge} = \rho_{eg}^*$$

$$\dot{\rho}_{gg} = \dot{c}_g c_g^* + c_g \dot{c}_g^* = -\frac{i}{2} \underbrace{\mathcal{N}^* c_e c_g^*}_{\rho_{eg}} - \frac{i}{2} \cancel{\delta c_g c_g^*} + \frac{i}{2} \underbrace{\mathcal{N} c_e^* c_g}_{\rho_{ge}} + \frac{i}{2} \cancel{\delta c_g^* c_g} = \frac{i}{2} (\mathcal{N} \rho_{ge} - \mathcal{N}^* \rho_{eg})$$

$$\dot{\rho}_{ee} = \dot{c}_e c_e^* + c_e \dot{c}_e^* = \frac{i}{2} \cancel{\delta c_e c_e^*} - \frac{i}{2} \underbrace{\mathcal{N} c_g c_e^*}_{\rho_{ge}} - \frac{i}{2} \cancel{\delta c_e^* c_e} + \frac{i}{2} \underbrace{\mathcal{N}^* c_g^* c_e}_{\rho_{eg}} = \frac{i}{2} (\mathcal{N}^* \rho_{eg} - \mathcal{N} \rho_{ge})$$

$$\dot{\rho}_{ge} = \dot{c}_g c_e^* + c_g \dot{c}_e^* = -\frac{i}{2} \underbrace{\mathcal{N}^* c_e c_e^*}_{\rho_{ee}} - \frac{i}{2} \underbrace{\delta c_g c_e^*}_{\rho_{ge}} - \frac{i}{2} \underbrace{\delta c_e^* c_g}_{\rho_{ge}} + \frac{i}{2} \underbrace{\mathcal{N}^* c_g^* c_g}_{\rho_{gg}} = \frac{i}{2} \mathcal{N}^* (\rho_{gg} - \rho_{ee}) - i \delta \rho_{ge}$$

$$\dot{\rho}_{eg} = \dot{c}_e c_g^* + c_e \dot{c}_g^* = \frac{i}{2} \underbrace{\delta c_e c_g^*}_{\rho_{eg}} - \frac{i}{2} \underbrace{\mathcal{N} c_g c_g^*}_{\rho_{gg}} + \frac{i}{2} \underbrace{\mathcal{N} c_e^* c_e}_{\rho_{ee}} + \frac{i}{2} \underbrace{\delta c_g^* c_e}_{\rho_{eg}} = \frac{i}{2} \mathcal{N} (\rho_{ee} - \rho_{gg}) + i \delta \rho_{eg} = \dot{\rho}_{ge}^* \checkmark$$

# Optical Bloch equations in density matrix form

$$(I) \dot{\rho}_{gg} = \frac{i}{2} (\mathcal{N} \rho_{ge} - \mathcal{N}^* \rho_{eg}) + \Gamma \rho_{ee}$$

$$(II) \dot{\rho}_{ee} = \frac{i}{2} (\mathcal{N}^* \rho_{eg} - \mathcal{N} \rho_{ge}) - \Gamma \rho_{ee} = -\left( \frac{i}{2} (\mathcal{N} \rho_{ge} - \mathcal{N}^* \rho_{eg}) + \Gamma \rho_{ee} \right) = -\dot{\rho}_{gg}$$

$$(III) \dot{\rho}_{ge} = \frac{i}{2} \mathcal{N}^* (\rho_{gg} - \rho_{ee}) - \left( \frac{\Gamma}{2} + i\delta \right) \rho_{ge}$$

$$(IV) \dot{\rho}_{eg} = \frac{i}{2} \mathcal{N} (\rho_{ee} - \rho_{gg}) - \left( \frac{\Gamma}{2} - i\delta \right) \rho_{eg}$$

Translation table from S to R

$$S = \frac{1}{2}(I + \vec{R} \cdot \vec{\sigma}) = \begin{pmatrix} S_{ee} & S_{eg} \\ S_{ge} & S_{gg} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}; \quad I = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}; \quad \sigma_x = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} & 1 \\ -i & \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$\begin{aligned} \frac{1}{2}(1+z) &= S_{ee} & z &= 2S_{ee} - 1 & \frac{1}{2}(x-iy) &= S_{eg} \\ \frac{1}{2}(1-z) &= S_{gg} & & & \frac{1}{2}(x+iy) &= S_{ge} \end{aligned}$$

$$\begin{aligned} 1 &= S_{ee} + S_{gg} & iy &= S_{ge} - S_{eg} & y &= -i(S_{eg} - S_{ge}) & \dot{y} &= i(\dot{S}_{eg} - \dot{S}_{ge}) \\ z &= S_{ee} - S_{gg} & x &= S_{eg} + S_{ge} & & & \dot{x} &= \dot{S}_{eg} + \dot{S}_{ge} \end{aligned}$$

$$\dot{z} = \dot{S}_{ee} - \dot{S}_{gg} = 2\dot{S}_{ee} = -2\dot{S}_{gg}$$

$\dot{S}_{ee} = -\dot{S}_{gg} \checkmark \Rightarrow$  (I) & (II) consistent

Translation from  $\mathcal{N}, \mathcal{N}^*$  to  $\mathcal{N}_R, \mathcal{N}_I$

$$\mathcal{N} = \mathcal{N}_R + i\mathcal{N}_I$$

$$\mathcal{N}^* = \mathcal{N}_R - i\mathcal{N}_I$$

$$\mathcal{N}_R = \frac{1}{2}(\mathcal{N} + \mathcal{N}^*)$$

$$\mathcal{N}_I = -\frac{i}{2}(\mathcal{N} - \mathcal{N}^*)$$

Translation of opt. Bloch eqn. from density matrix form to Bloch vector form

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$$(II) \quad \frac{\gamma}{2} \dot{z} = \frac{i}{2} \left( (\mathcal{N}_R - i\mathcal{N}_I) \frac{\gamma}{2} (x - iy) - (\mathcal{N}_R + i\mathcal{N}_I) \frac{\gamma}{2} (x + iy) \right) - \Gamma \frac{\gamma}{2} (1+z)$$

$$\begin{aligned} \dot{z} &= x \left[ \frac{i}{2} (\mathcal{N}_R + i\mathcal{N}_I) - \frac{i}{2} (\mathcal{N}_R + i\mathcal{N}_I) \right] \\ &\quad + y \left[ +\frac{\gamma}{2} (\mathcal{N}_R - i\mathcal{N}_I) + \frac{\gamma}{2} (\mathcal{N}_R + i\mathcal{N}_I) \right] \\ &\quad - (1+z) \Gamma \\ &= x (+\gamma) \mathcal{N}_I + y (+\gamma) \mathcal{N}_R - (1+z) \Gamma \end{aligned}$$

$$(III) \quad \frac{\gamma}{2} (\dot{x} + i\dot{y}) = -\frac{i}{2} (\mathcal{N}_R - i\mathcal{N}_I) z - \left( \frac{\Gamma}{2} + i\delta \right) \frac{\gamma}{2} (x + iy)$$

$$(IV) \quad \frac{\gamma}{2} (\dot{x} - i\dot{y}) = +\frac{i}{2} (\mathcal{N}_R + i\mathcal{N}_I) z - \left( \frac{\Gamma}{2} - i\delta \right) \frac{\gamma}{2} (x - iy)$$

$$\begin{aligned} (III) + (IV): \quad \dot{x} &= x \left[ -\frac{\gamma}{2} \left( \frac{\Gamma}{2} + i\delta \right) - \frac{\gamma}{2} \left( \frac{\Gamma}{2} - i\delta \right) \right] \\ &\quad + y \left[ -\frac{i}{2} \left( \frac{\Gamma}{2} + i\delta \right) + \frac{i}{2} \left( \frac{\Gamma}{2} - i\delta \right) \right] \\ &\quad + z \left[ -\frac{i}{2} (\mathcal{N}_R - i\mathcal{N}_I) + \frac{i}{2} (\mathcal{N}_R + i\mathcal{N}_I) \right] \\ &= x \left( -\frac{\Gamma}{2} \right) + y (\delta) + z (i\mathcal{N}_I) \end{aligned}$$

$$\begin{aligned} (III) - (IV): \quad i\dot{y} &= x \left[ -\frac{\gamma}{2} \left( \frac{\Gamma}{2} + i\delta \right) + \frac{\gamma}{2} \left( \frac{\Gamma}{2} - i\delta \right) \right] \Rightarrow \dot{y} = x (-\delta) + y \left( -\frac{\Gamma}{2} \right) + z \left( \frac{i}{2} \mathcal{N}_R \right) \\ &\quad + y \left[ -\frac{i}{2} \left( \frac{\Gamma}{2} + i\delta \right) - \frac{i}{2} \left( \frac{\Gamma}{2} - i\delta \right) \right] \\ &\quad + z \left[ -\frac{i}{2} (\mathcal{N}_R - i\mathcal{N}_I) + \frac{i}{2} (\mathcal{N}_R + i\mathcal{N}_I) \right] \\ &= x (-i\delta) + y \left( -i\frac{\Gamma}{2} \right) + z \left( -\frac{i}{2} \mathcal{N}_R \right) \end{aligned}$$

Ord block eqn in vector form

$$\dot{x} = -\frac{\Gamma}{2}x + \delta y - \nu_L z$$

$$\dot{y} = -\delta x - \frac{\Gamma}{2}y - \nu_R z$$

$$\dot{z} = -\nu_L x + \nu_R y - (1+z)\Gamma$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & \delta & -\nu_L \\ -\delta & 0 & -\nu_R \\ -\nu_L & \nu_R & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \frac{\Gamma}{2} \begin{pmatrix} x \\ y \\ -2(1+z) \end{pmatrix}$$

Linearized eqn

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\mathbf{z} + \mathbf{c}(1+z)\Gamma$$

$$\mathbf{A} = \begin{pmatrix} 0 & \delta & -\nu_L \\ -\delta & 0 & -\nu_R \\ -\nu_L & \nu_R & 0 \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} -\nu_L \\ -\nu_R \\ 0 \end{pmatrix}$$

$$\mathbf{c} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$\mathbf{A} + \mathbf{b}\mathbf{z} + \mathbf{c}\Gamma = \begin{pmatrix} 0 & \delta & -\nu_L \\ -\delta & 0 & -\nu_R \\ -\nu_L & \nu_R & 0 \end{pmatrix} - \frac{\Gamma}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\mathbf{A} + \mathbf{b}\mathbf{z} + \mathbf{c}\Gamma = \begin{pmatrix} 0 & \delta & -\nu_L \\ -\delta & 0 & -\nu_R \\ -\nu_L & \nu_R & 0 \end{pmatrix} - \frac{\Gamma}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Check of cross-product form of opt. Bloch eqn.

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$$\frac{\partial}{\partial t} \vec{R} = \vec{\Omega} \times \vec{R}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \Omega_y z - \Omega_z y \\ \Omega_z x - \Omega_x z \\ \Omega_x y - \Omega_y x \end{pmatrix} = \begin{pmatrix} -\Omega_E z + \delta y \\ -\delta x + \Omega_R z \\ -\Omega_z x + \Omega_R y \end{pmatrix} \begin{array}{l} \rightarrow \Omega_y = -\Omega_E \quad ; \quad \Omega_z = -\delta \\ \rightarrow \Omega_z = -\delta \quad ; \quad \Omega_x = +\Omega_R \\ \rightarrow \Omega_x = \Omega_R \quad ; \quad \Omega_y = -\Omega_E \end{array} \quad \checkmark$$

$$\Rightarrow \vec{\Omega} = \begin{pmatrix} -\Omega_R \\ -\Omega_E \\ -\delta \end{pmatrix} = \begin{pmatrix} |\Omega| \cos \varphi_L \\ -|\Omega| \sin \varphi_L \\ -\delta \end{pmatrix}$$

$$\Omega = |\Omega| e^{-i\varphi_L} = |\Omega| \cos(\varphi_L) + i|\Omega| \sin(-\varphi_L) = \underbrace{|\Omega| \cos \varphi_L}_{\Omega_R} + i \underbrace{(-|\Omega| \sin \varphi_L)}_{\Omega_E}$$